# Seki with 2 Liberties per Chain in the game of Go

Thomas Wolf
Department of Mathematics & Statistics, Brock University

December 19, 2016

**Abstract**

Seki positions of the game of Go that are discussed in this paper involve only chains with 2 liberties with the minor generalization of what will be called 'negligible' 1- or 2-stone chains with 1 liberty sitting in a 1-chain eye. There are no kos in the seki and no kos can result from play in the seki. Any move, other than capturing a negligible chain or filling a 1-point eye composed of multiple chains of same colour and thus linking these chains, would result in the instant capture of the placed stone, and its attached chain of the same colour. Also, the captured chain must be large enough, and of the right shape, such that the captured player then loses, without even the possibility of any type of ko, like 'mannen ko'. Therefore, 'throw-in ko', and sacrifices of 'Nakade' shapes, and positions like 'bent 4 in the corner', are excluded. We call the remaining, simplest, permitted positions Basic Seki.

The article starts with describing deformations of seki positions that do not change the essential structure of a seki which is summarized in a so-called Basic Seki Graph. By using such graphs it is shown that all chains which share some liberty, say $A$, can have in total at most two more liberties other than $A$. As a consequence Basic Seki have either a linear or a circular topology.

Furthermore, if the position has $> 2$ empty points, then there can be at most 3 chains which have the same 2 empty points as liberties and, moreover, these chains need to have the same colour.

These properties allow us to recognise Basic Seki, after 1) verifying that sacrificing any chain leads to an unconditional loss, and then 2) verifying the above properties of Basic Seki. These properties are all local, so that the total effort goes linearly with the number of liberties and chains that are involved if that number gets large.

Based on the necessary and sufficient properties of Basic Seki, a numerical encoding of any Basic Seki is possible that allows a straightforward algorithm for formulating all possible topologies of Basic Seki and thus for creating any Basic Seki position. An approximate formula for the number of different topologies of Basic Seki has been derived.

Finally, examples of modifications of seki positions are shown that change their Basic Seki Graph. The resulting positions are used to illustrate the numerical encoding of their topology.

# 1 Introduction

A prerequisite for playing a game is to know when to stop. In the game of Go one possible terminal position is called *seki* which is a Japanese term adopted to English which means mutual life. In contrast to unconditional life, in seki positions two opponent chains share one or more liberties which neither side can occupy without being captured. Diagrams 1 and 2 show such positions. A study of final positions is also of interest for Computer Go. A recent analysis of liberty races (known as semeai) performed in [2] considers races between two essential chains that may require approach moves for their capture. Before one will be able to generalize the analysis of semeai to more than two essential chains, at least all possible types of final positions should be known. This paper characterizes the simplest class of terminal seki positions where each essential chain has exactly two liberties and is not able to get even temporarily one more. If either side were to make a move on a shared liberty in these seki, then this chain would instantly be captured by the opponent. For these reasons we call them *Basic Seki*. In other seki, chains are more resistant, for example, by either having more than two liberties or by having the opportunity to obtain at least temporarily one or more liberties through capture.

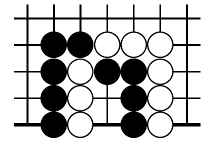

Diagram 1.

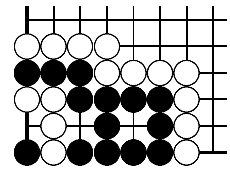

Diagram 2.

This article is the first in an attempt to classify seki positions. In a follow up text [4], a different simplifying assumption is made. It considers positions where each liberty has only one or two chains as neighbours.

Before classifying Basic Seki in this paper according to their topological type we will show variations that do not change the structure of the seki. This is done in the following section. The type of seki that we consider is defined in section 3 together with a special type of graphs that represents them. The revealed simple topology of these graphs does allow a compact numerical encoding that is introduced in section 4. Examples for modifications of seki that change the number of essential chains and thus the topology are shown in section 5. The appendix contains an overview of all topologies of Basic Seki with up to four chains.

# 2 Equivalent Seki

In a classification of seki we will not distinguish between positions that are only slightly modified. We therefore look in the following section at first at deformations that do not change the essence of a seki position.

## 2.1 Deformations

Basic Seki discussed in this article involve chains that have exactly two liberties of which at least one is shared with other chains. These are called *essential chains* because their capture would convert the seki into unconditional life of the neighbouring opponent chains. Apart from these essential chains there can occur only surrounding chains which are unconditionally alive and which do not share a liberty with an essential chain or further 1-stone or 2-stone chains which are situated in 1-liberty eyes like the black stone in the corner of Diagram 2. Such a 1-stone chain is irrelevant for the seki.[1]

Because the capture of an essential chain would instantly change the nature of the position it does not matter how large such an essential chain is or which shape it has *as long*

[1]If it would be a 2-stone chain then its capture could be used as a ko-threat. But each seki can be used as a threat in a large game deciding ko so ko is not relevant in a classification of seki.

*as the position is still a seki.* In this section we will illustrate which deformations preserve the structure of a seki and which even destroy a seki.

For example, when in Diagram 3 the black 3-stone chain is reduced to 1 stone then the position in Diagram 4 is not a seki anymore. Similarly, when in Diagram 5 the white 4-stone chain is reduced to 3 stones the position in Diagram 6 is not a seki anymore whereas in Diagram 7 the white chain is essential, even though it also has only 3 stones. The difference is that these 3 white stones maintain a cross cut, i.e. a configuration like  .

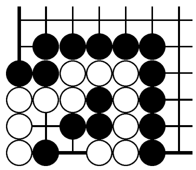

Diagram 3.

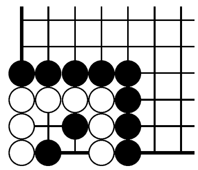

Diagram 4.

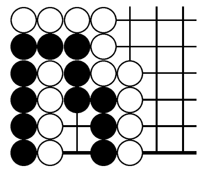

Diagram 5.

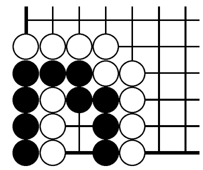

Diagram 6.

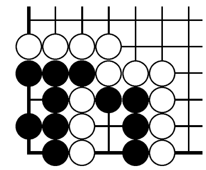

Diagram 7.

For Basic Seki, all that matters is which chains participate in which liberty. Therefore one can change the size and shape of (essential) chains (as long as they are not shrunk to an extend that they can be sacrificed and the position is not a seki anymore) without changing the nature (topology) of the seki-position. We will call these changes *deformations* of a position. In those quasi continuous deformations the so-called 'Common Fate Graph' (CFG) does not change (see [3], where the term 'Common Fate Graph' was introduced). This graph is obtained by representing each chain and each empty point on the board by a node (dot) and by linking two nodes with an edge (line) if both are neighbours to each other. For example, deformations which do not change this graph include a motion of the position to the edge of the board or a motion away from the edge of the board. In this sense the following Diagrams 8 - 12 are all fully equivalent.

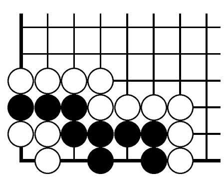

Diagram 8.

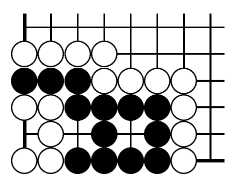

Diagram 9.

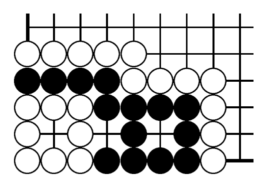

Diagram 10.

All five positions in Diagrams 8 - 12 have the same common fate graph shown in Figure 1

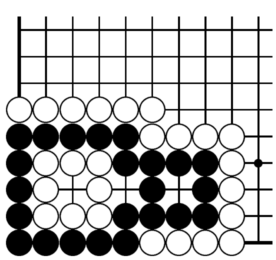

Diagram 11.

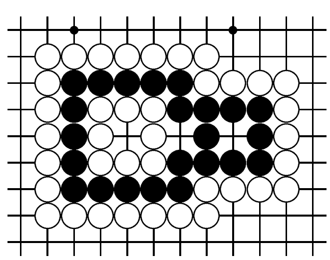

Diagram 12.

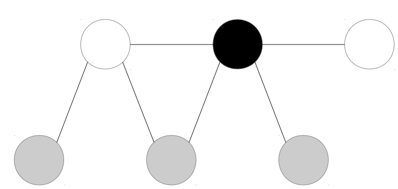

Figure 1:
The corresponding CFG

where the nodes in the top row represent the three chains (a red node for a white chain and a blue node for a black chain) and the nodes underneath correspond to the three liberties.

## 2.2 Introducing Cross Cuts

The following sequence of Diagrams shows a deformation from Diagram 13 to 14, then the introduction of a cross cut from Diagram 14 to 15 and a deformation which allows us to shift the cross cut. Although these changes look more substantial, they also do not change the structure of the seki.

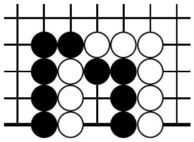
Diagram 13.

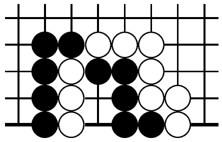
Diagram 14.

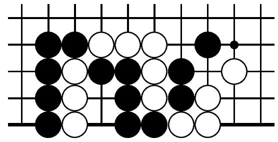
Diagram 15.

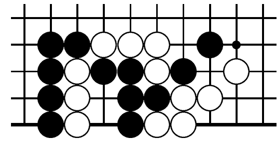
Diagram 16.

In the following example, the position in Diagram 17 is deformed and two crosscuts are introduced to obtain Diagram 18 which after further deformation and the removal of a crosscut takes the form in Diagram 19.

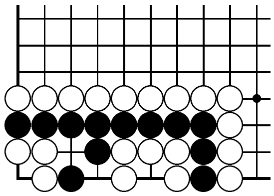
Diagram 17.

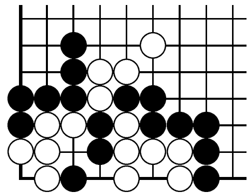
Diagram 18.

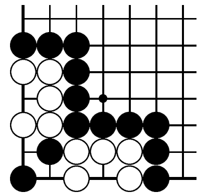
Diagram 19.

The introduction of a cross cut changed the number of chains and thus the common fate graph, but the nature of the seki has not changed. Thus a different graph with fewer edges and nodes should be used if all chains are essential, i.e. if the capture of any one chain implies that all chains of that colour are dead. Before introducing this new type of graph, we define the type of seki that we investigate.

Although the main result of the paper that Basic Seki can only have a linear or circular topology is very descriptive, we choose a formal approach to obtain this result[2].

# 3 Basic Seki

As the name suggests, Basic Seki are the most easily recognizable type of seki. They are also the most fragile seki where a stone put on a liberty shared by black and white chains leads to instant capture of that stone together with its chain.

## 3.1 Some Definitions

In the following we define different types of chains and seki.

**Def.**: *A chain is called 'negligible' if it*

- *is the only chain within an opponent 1-chain 1-liberty eye* (that is made up from only one opponent chain with only one inner empty point and therefore the negligible chain has only one liberty and this liberty is the only empty point within the eye), *and*
- *consists of less than 3 stones.*

An example for a negligible chain is the single black stone in the lower left corner of Diagram 2. If such a negligible chain has two stones and would be captured then it would be replaced by one stone in the next move. In any case the capturing chain did not gain any liberty through the capture.

**Def.**: *'Seki' are positions with the two properties:*

- *If any one side is allowed to place stones repeatedly then it can capture all chains of the other side.*

[2]as partially suggested by referees

- *If both sides alternate in their moves then neither side can capture a non-negligible chain without losing more of its own stones than it can capture.*

**Def.**: *A chain as part of a seki is called 'essential' if it's capture implies that all chains of the same colour in the seki can be captured.*

In Diagrams 1 to 19 except Diagrams 4, 6 which are not seki, all chains other than surrounding chains with outside liberties and other than the single negligible chain in Diagram 2 are essential chains.

**Def.**: *A 'Basic Seki' is a seki that involves only*

- *essential chains with each having exactly two liberties and possibly negligible chains,*
- *empty points that are a liberty of at least one essential chain and possibly a negligible chain but not a liberty of an unconditionally alive chain.*

Examples of Basic Seki have been shown above in Diagrams 8 to 19.

## 3.2 Comments on Basic Seki

The following are comments on the implications of the above definitions.

- Although ko is not explicitly mentioned, the definitions imply that ko can not come up in a Basic Seki because the only chains that are not essential are within 1-chain 1-liberty eyes.

- In Basic Seki an empty point can only be of the following four types:
  1. a liberty shared by chains of both colours,
  2. a liberty in a 1-chain 1-liberty eye,
  3. a liberty without an empty neighbouring point shared by chains of only one colour (i.e. a liberty in a multi-chain 1-liberty eye),
  4. a liberty with an empty neighbouring point where all chains neighbouring these two empty points having the same colour.[3]

- Basic Seki as defined above are terminal positions in the following sense. Liberties shared between essential chains and unconditionally alive chains are excluded. Also, moves on liberties shared by essential chains of different colours would lead to instant capture and are not possible either.

  Moves on liberties of type 2 and 3 above are possible by one of both players but are useless. Excluding seki with such liberties would not allow more detailed characterizations for such a more narrow class of seki. We therefore allow Basic Seki to have liberties of types 3 and 4 and allow negligible chains.

## 3.3 Basic Seki Graphs

When dealing with Basic Seki the CFGs still contain irrelevant information. Negligible chains should not appear in a seki graph. Furthermore, neighbourhood relations between essential chains should also not be shown. Because in Basic Seki all chains are essential or negligible, it does not matter which essential chain is attached to which other essential chain. If one of them is caught then all chains of this colour are dead and all others are alive.

[3]If, for example, one of both empty points would have only white neighbouring chains and the other only black neighbouring chains then both sides could start a ko. Kos are not covered in this paper.

In Basic Seki each essential chain has 2 liberties. Similarly, in a graph each edge has 2 end points. We therefore will represent Basic Seki by what we will call a *Basic Seki Graph* (BSG) where each chain is an edge in the graph and each empty point is a node in the graph. Because the colour of a chain does matter, edges need to be coloured, for example, we will draw black chains as blue dotted edges and white chains as red solid edges.

Furthermore, a liberty is either shared only by essential chains or it is the only liberty within a 1-chain 1-liberty eye (with possibly a negligible chain inside). Liberties are not shared between an essential chain and a surrounding unconditionally alive chain because the unconditionally alive chain could be extended to also occupy this liberty without risk but with a possible impact on the essential chain. We do not lose generality by considering only positions without such liberties. If occupying these liberties does still result in a Basic Seki then the position is considered, otherwise it does not fall under the scope of this paper.

To summarize, there is a one-to-one correspondence between each red solid line, blue dotted line and node in the BSG and each white essential chain, black essential chain and liberty of an essential chain in the seki.

For the position in Diagram 20 the Common Fate Graph (CFG) and the more compact Basic Seki Graph (BSG) are shown in Figure 2. Inside each node are the coordinates of a stone identifying the corresponding chain or the coordinates of the corresponding empty point.

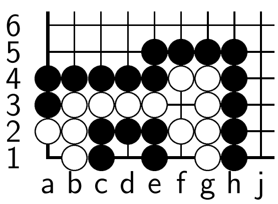


Diagram 20.

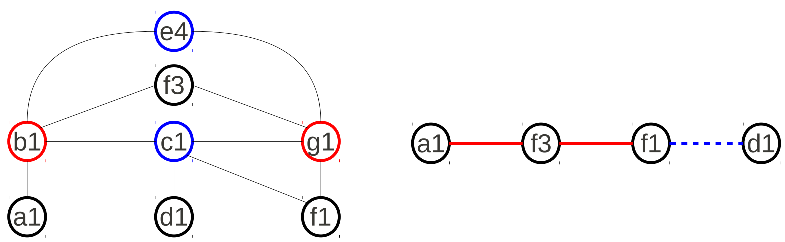


Figure 2: The corresponding CFG and BSG

Although every Basic Seki position can be associated with a graph, not every graph can be associated with a Basic Seki. In the following section we collect necessary conditions on, what we will call, *admissible* graphs that represent Basic Seki. This will allow us to classify Basic Seki by generating all possible admissible graphs, i.e. all topological types of Basic Seki.

## 3.4 Necessary Conditions for a Graph to represent a Basic Seki

In this section we collect necessary conditions for a BSG to represent a Basic Seki. The following section summarizes them and in section 3.5 it is shown that the set of conditions is complete, i.e. that for each graph satisfying these conditions, a corresponding Basic Seki can be constructed. In the following figures white chains are represented as red solid lines, black chains as dotted blue lines and liberties as nodes as done on the right in Figure 2. These are the conditions.

1. *At least one node has to be shared between at least one red and one blue edge* as otherwise all liberties would be eyes and opponent moves in these eyes would be suicide because each chain participates in 2 1-liberty eyes and therefore all white chains would be unconditionally alive and all black chains would be unconditionally alive.

2. *Each node has at least one and at most four edges*, i.e. $1 \leq$ (degree of node) $\leq 4$ because if a node had no edge and therefore was surrounded only by empty points then each side could put a stone on that field without taking a liberty of a chain and then the node could not be part of a terminal seki.

3. *There can not be four edges of one colour between two nodes* because according of condition 2 there would be no other edge emerging from these two nodes and therefore there would also be no other nodes and no other edges in this position which would contradict condition 1.

4. *The BSG is planar*, i.e. it can be displayed in the plane without crossing edges. The reason is that a Go board is of course 2-dimensional and Go stones do not lie on top of each other.

5. *The BSG has no loops*, because a single empty point can be only one liberty for a chain. It does matter which chains are neighbours of an empty point but not on how many sides each chain neighbours an empty point.

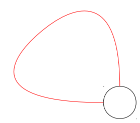

Figure 3: A loop that does not occur in a BSG

6. *If two edges of same colour*, say red, *end in a node*, say $M$, *and at least one blue edge also ends at $M$ then both red edges must have their other end in the same other node*, say $N$. If that would not be the case then White making a move on $M$ would join both chains and this new chain would still have two liberties whereas the black chain that was also neighbour to the mixed node $M$ would be under atari. This clearly would not be a terminal seki position. An example is given in Figure 4.

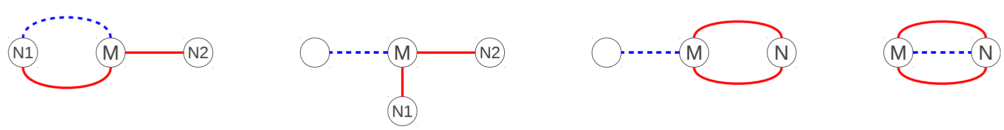


Figure 4: Two forbidden and two admissible graphs

7. *If two nodes are linked to each other by edges of different colour then these two nodes are the only possible nodes of the admissible graph.* If there would be another node in the BSG like node N2 in the left graph of Figure 4 then there needs to be at least one edge of some colour, say red, to one of the two linked nodes (N1 and M), say to M, and then there would be at least 2 nodes of one colour (red) ending at node M which connect to 2 different nodes (N1,N2) and M would also be an endpoint of a blue edge. This violates condition 6.

8. *If a node has edges of only one colour then these edges may reach at most two other nodes*, i.e. it is not possible that a node has only 3 red edges which connect to 3 different nodes because White occupying this liberty would join all 3 chains and generate a chain with 3 liberties which could attack opponent chains with only 2 liberties. An example of a forbidden and an admissible graph is given in Figure 5.

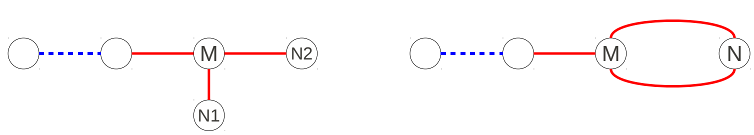


Figure 5: A forbidden and an admissible graph

9. *Edges originating from one node can reach at most two other nodes.* If three or more nodes would be reached from a node $M$ then because of condition 8 above the originating edges can not all be of the same colour.

   But the originating edges can also not be of different colour. If there is only one red edge then this can reach only one other node. If there are more red edges then they can also reach only one other node because of condition 6 above. The same is true for blue edges. Therefore, if red and blue edges would originate from $M$ then at most 1+1=2 other nodes could be reached and not 3.

   This proves that each node is linked to at most 2 other nodes.

   Furthermore, if a node has 2 neighbouring nodes then each one of them can be reached only by edges of the same colour (see condition 6).

### 3.5 Summary of Necessary Conditions on Basic Seki Graphs

Based on the necessary conditions in section 3.4, especially condition 9, BSG have a simple structure. A BSG

1. is planar, has at least one red edge and one blue edge, no loop, only nodes with degree $\leq 4$ (at most 4 edges emerging from any node), and
2. (a) either consists of only two nodes (which are consequently linked by at least one red and one blue edge), or

   (b) consists of $> 2$ nodes where each node is linked to at most two nodes and each one of them is only linked by $\leq 3$ edges of the same colour. As a consequence the whole sequence of nodes has either linear or circular topology. If it is linear then necessarily the first and last node have only edges of one colour attached which all lead to one other node.

   The case 2a of only two nodes is seen in the rest of the paper as a limit of the circular type of case 2b with the smallest possible circle being made from two nodes.

### 3.6 Sufficient Conditions on Basic Seki Graphs

In this section we establish that the above constraints are not only necessary but also sufficient.
**Lemma:** *Any graph satisfying the constraints of section 3.5 represents a Basic Seki.*
**Proof:**

1. Because of point 1 in section 3.5 the lines of an enlarged printout of the graph could be represented through black and white chains on a large Go-board. Empty space between edges of different colour would be filled by stones of either side and empty space between edges of same colour would be filled by a living chain of the opposite colour. Then all chains would have exactly 2 liberties.
2. If the graph has only 2 nodes (case 2a above) then any side making a move on any one of them produces a new chain that has only one liberty which can be captured.
3. If the graph has $> 2$ nodes (case 2b above) then

   (a) if a node A has only one neighbouring node B (linked necessarily by edges of only one colour, say red) then a move of White on A results in a single white chain with only 1 liberty (B) which can be captured in the next move,

(b) if a node A has two neighbouring nodes B,C linked to A by edges of only one colour, say red, then only White can make a move on A resulting in a single white chain with two liberties B,C as it should be the case for all chains to have 2 liberties in a Basic Seki,

(c) if a node A has a neighbouring node B linked by only red edges and a neighbouring node C linked by only blue edges then either side making a move on A produces a chain with only 1 liberty that can be captured in the next move.

□

**Theorem:** *The conditions listed in section 3.5 are necessary and sufficient for a graph to represent a Basic Seki and thus be a BSG.*
**Proof:** Because of the above lemma the set of necessary conditions is complete, i.e. these conditions are also sufficient. □

All BSG with up to 4 edges are given in the appendix.

### 3.7 Some Examples for Necessary Conditions to be Complete

According to condition 3 in section 3.4 it is not possible that four edges of one colour link 2 nodes because all chains would be unconditionally alive. But it is possible that three edges of one colour link two nodes as shown in Diagram 21 or that one node has four edges of one colour as shown in Diagram 22.

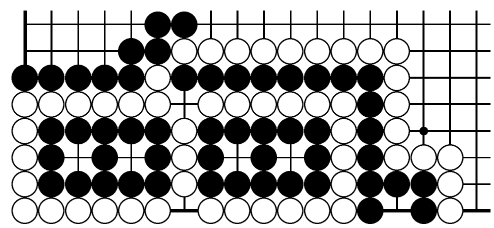

Diagram 21.

Examples of two edges of one colour linking the same two nodes are given through the following situations:

- the two chains are diagonal neighbours separated by the two liberties like in Diagrams 18 and 19, or
- the two liberties which correspond to both nodes are sitting next to each other on the board, like in Diagram 68, or
- an essential chain of the other colour (Black) lies between both, like in Diagrams 69, 70, or
- a living chain of the other colour (Black) separates the two (white) chains, like in Diagram 71.

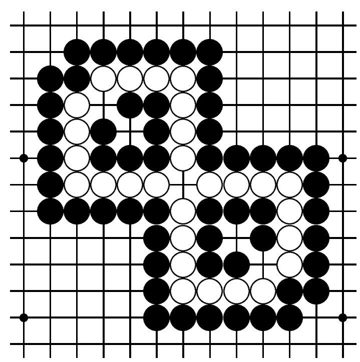

Diagram 22. “No 8”

### 3.8 BSG Generators

In this section we present a generation method to create all BSG through a repeated substitution of edges as shown in Figure 6 and their colour inverted versions applied to seed BSG shown in Figure 7. We established in the previous section that BSG have either a circular or linear structure, i.e. that a basic seki has either zero or two 1-chain eyes. Therefore the most elementary BSG are shown in Figure 7 with sekis in Diagrams 1, 2 being examples for them.
The following restrictions apply to the left two nodes in Figure 6 in order to be a substitution converting a BSG to another BSG.

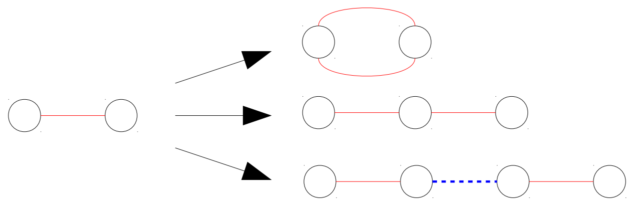

Figure 6: Generators for BSG

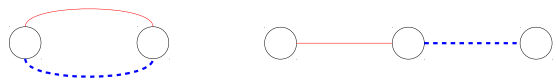

Figure 7: Seeds of BSG Generation

- For the top substitution each node of the left must have at most 3 edges ending in it so that after adding another edge each node has not more than 4 edges ending in it (condition 2).
- For the middle and bottom substitutions to create a BSG both nodes are either not linked to each other through another edge or they are linked to each other through one or more blue edges but not through another red edge. If they would be linked through another red edge then after the substitution they would each be linked to two different neighbouring nodes through a red edge and therefore could not have a blue edge ending in them according to condition 6 and could also not have been linked to another node by a red edge according to condition 8 and thus the BSG could not have had any blue edges which would mean that it was not a BSG before the substitution.

Starting from the two seeds in figure 7 one can apply the middle and bottom substitution as often as one wants to get a linear or circular BSG of any size and any sequence of red and blue edges and then use the top substitution to add parallel edges of same colour. In this way, any BSG can be generated from the 2 seeds, by repeated use of combinations of the 3 substitutions.

# 4 A Numerical Encoding

Because of the simple topological structure of Basic Seki uncovered in section 3.5 we are able to characterize each one with a sequence of digits.

## 4.1 The Translation Rules

Let us introduce the natural abbreviations

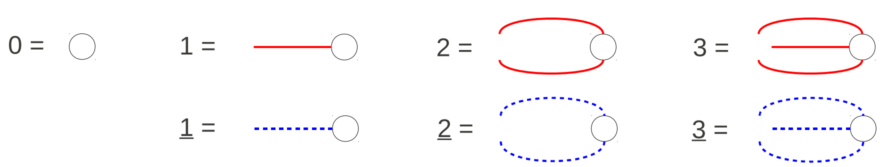


Figure 8: Abbreviations of graph elements by digits

The BSG is either a linear or a circular sequence of nodes linked pairwise by 1,2 or 3 edges of same colour. If the BSG is a linear sequence then the numerical encoding starts with a 0. The following are examples of seki of linear type with their numerical encoding. The corresponding BSG or colour inverted version can be found in the appendix.

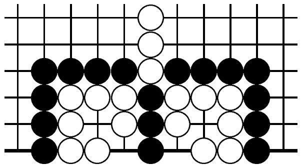

Diagram 23.
with encoding $01\underline{1}1$

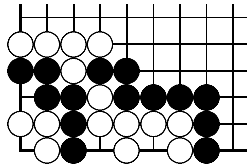

Diagram 24.
with encoding $01\underline{1}1$

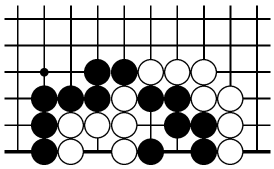

Diagram 25.
with encoding $01\underline{2}$

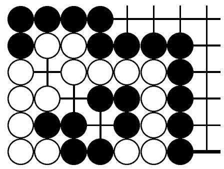

Diagram 26.
with encoding $02\underline{2}$

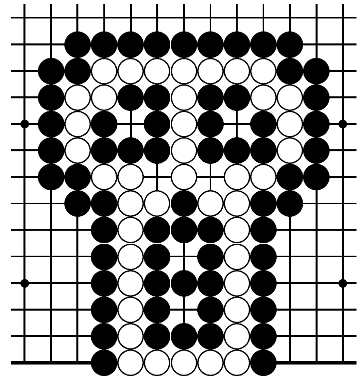

Diagram 27.
"The Scream"
with encoding $0\underline{1}2\underline{1}$

Apart from the initial 0 the order of the remaining digits can be inverted for a linear graph.

A circular graph will be encoded by *not* having a 0 as the first digit. The following are examples.

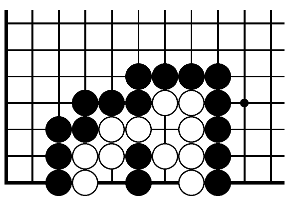

Diagram 28.
with encoding $\underline{1}11$

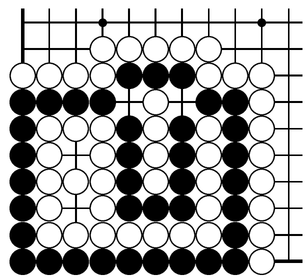

Diagram 29.
with encoding $\underline{3}1$

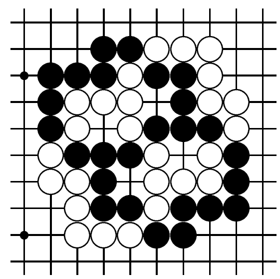

Diagram 30.
with encoding $\underline{1}1\underline{1}1$

Although the seki in Diagram 28 has a multi-chain eye, it is a circular seki. The reason is that the node in the BSG that corresponds to the eye has two neighbouring nodes in the graph. In other words, the two white chains that build the eye have two different second liberties. If they would have the same second liberty as in Diagram 26 then the seki would be of linear type.

The seki in Diagram 29 has 2 liberties and by convention it belongs to the circular class too. This convention is made because it allows us to use the numerical encoding: chains of one colour link a first liberty with the second one and chains of the other colour linking the second liberty with the first one, i.e. the numerical encoding consists of one underlined digit and one digit that is not underlined.

We will have two more rules that compact and generalize the notation. First, by treating sequences of edges like a product one can abbreviate repetitions through an exponent, like $1\underline{2}1\underline{2} = (1\underline{2})^2$. Second, to indicate that two seki are attached to each other on the board but their BSG are disconnected, we will append the two encodings through a ; sign. Examples are given in section 5.1.

In a numerical encoding the number of black chains is given as the sum of all underlined digits and the number of white chains is given as the sum of not underlined digits (each multiplied by their respective exponent). The number of liberties in the seki is given through the total number of digits, the leading zero for linear seki included.

## 4.2 The Generation of all Topological Types of Basic Seki

Based on the observations above one obtains all possible Basic Seki encodings up to a switch of colours by

- optionally starting with a 0 to encode a seki with linear topology,
- having apart from the optional initial 0 an arbitrary sequence of digits 1,$\underline{1}$,2,$\underline{2}$,3,$\underline{3}$ with the only conditions that
  - at least one underlined digit and one not underlined digit need to come up somewhere in the sequence (otherwise the position is alive),
  - the sum of any two neighbouring digits is not bigger than 4 and in the case of a circular seki then in addition the sum of the first and last digit is also not bigger than 4 (because a point on the Go-board has at most 4 neighbours),
- avoiding identical *linear* BSG through identifying two sequences that both start with a 0 and apart from the initial 0 one sequence inverted gives the other sequence or if one sequence colour switched (underlined $\leftrightarrow$ not underlined) and inverted gives the other sequence,
- avoiding identical *circular* BSG through identifying two sequences that both do not start with a zero if one sequence or its reverse (clockwise $\leftrightarrow$ counter clock wise) or its colour switched (underlined $\leftrightarrow$ not underlined) or its reversed and colour switched version is a cyclic permutation of the other sequence.

For any such sequence of digits one can construct a seki and any Basic Seki has such an encoding. Whether the seki would fit onto a standard $19 \times 19$ Go board is another question.

## 4.3 The Number of all Topological Types of Basic Seki

If one is curious about the number of all topologies of Basic Seki with in total $n$ liberties then one needs to compute the number of sequences that satisfy the conditions in section 4.2 above.

Let $n$ be the number of liberties in the seki. We deal with linear topologies first, i.e. the first digit in the sequence of $n$ digits is 0. Let us for now assume that we have only non-underlined digits 1,2,3 (only white chains) such that the sum of two successive digits is not more than 4 (one of the conditions above). Let $a_n$ be the number of such sequences of $n$ digits and let $b_{n,1}, b_{n,2}, b_{n,3}$ be the number of sequences of $n$ digits that end with a 1, 2 or 3. Consequentially it is $a_n = b_{n,1} + b_{n,2} + b_{n,3}$.

We then have $a_1 = 1$ (the sequence $\{0\}$), $a_2 = 3$ (the sequences $\{01\}$, $\{02\}$, $\{03\}$) i.e. $b_{2,1} = 1, b_{2,2} = 1, b_{2,3} = 1$.

When a sequence ends with the digit $i$ then by extending this sequence the next digit can be $1, ..., 4 - i$. This gives us the recursive formulas

$$\begin{aligned} b_{n+1,1} &= b_{n,1} + b_{n,2} + b_{n,3} \quad (= a_n), && (1) \\ b_{n+1,2} &= b_{n,1} + b_{n,2}, \\ b_{n+1,3} &= b_{n,1}. && (2) \end{aligned}$$

By adding up the left and right hand sides we get

$$\begin{aligned} a_{n+1} &= 2(b_{n,1} + b_{n,2} + b_{n,3}) - b_{n,3} + b_{n,1} \\ &= 2a_n - b_{n-1,1} + a_{n-1} && \text{(using (2), (1))} \\ &= 2a_n + a_{n-1} - a_{n-2} && \text{(using (1))} \end{aligned}$$

This recursive formula together with $a_1 = 1, a_2 = 3$ defines an exponentially fast growing sequence $a_n$ known as A006356 in the on-line encyclopedia of integer sequences. More details are found on [7].

If we not only have digits 1,2,3 but also allow $\underline{1}$,$\underline{2}$,$\underline{3}$, then we simply have to multiply $a_n$ by $(2^{n-1}-2)/2 = 2^{n-2}-1$ where $2^{n-1}$ comes from the $n-1$ options to underline a digit or not (except the first 0), $-2$ comes from the requirement not to have only non-underlined digits (not only white chains) and not only underlined digits (not only black chains) and 1/2 comes from the equivalence of two BSG when inverting the order of all digits. The factor 1/2 is not 100% correct because sequences may be symmetric wrt. inversion but for $n \to \infty$ the factor goes to 1/2.

In the case of circular topology, one also has to divide by a factor of $n$ which is the number of cyclic permutations (at least for larger $n$ whereas for small $n$ inversions and cyclic permutations may give occasionally the same sequence.)

What is the practical consequence of the exponentially fast growing number of topologies of Basic Seki? Due to results in this paper there is no need to store a table of such topologies.

The theorem in section 3.6 says that the 2 conditions in section 3.5 are all that need to be checked in order to verify that a position is a Basic Seki. At first one has to verify that no chain can be sacrificed like in Diagrams 4 and 6 which is a local check. Then one writes down the Basic Seki Graph and checks the *local* conditions in section 3.5. And all this effort goes linearly with $n$.

In the following section examples of BSG and numerical encodings are given. It also shows modifications of seki positions that change their BSG because they increase the number of chains while preserving the seki status.

# 5 Complicating a Seki

In this section we give examples of techniques how a given seki position can be modified into a new similar seki position with one more chain, i.e. how one can generate increasingly complicated seki positions. These changes also alter the BSG.

We start with an example of how to assemble a bigger seki from smaller ones.

## 5.1 Attaching Seki

Seki positions can be arbitrarily attached to each other if each chain has liberties in only one of the seki. The BSG of the composed seki consists of the disconnected graphs of the individual Basic Seki. For example, the seki in Diagram 31 published by Ger Hungerink [5] contains 128 chains but its BSG shows that the full board position consists of three separate Basic Seki: one in the upper left corner (the BSG is shown in Figure 9), one in the lower right corner (with same BSG, only colours switched) and one on the rest of the board.

This is not quite the same as three fully separate seki on the board. The three Basic Seki in Diagram 31 depend on each other. Any move in the inner large (elementary) seki would resolve one of the two corner seki. For example, a move of Black would imply all black chains of the center seki to be caught and therefore also in the upper left corner. A move of White in the center would turn also the lower right corner into Black territory. The BSG of the whole board consists of three disconnected sub graphs and has the encoding $\underline{1}2\underline{1}1;0(\underline{2}2)^4\underline{1}(2\underline{2})^6 1\underline{1}2\underline{1}1(\underline{2}2)^6\underline{1}1\underline{2}1\underline{1}(2\underline{2})^6 1(\underline{2}2)^4;\underline{1}1\underline{2}1$. The two ; signs separate the circular seki in the upper left corner (with its encoding starting at a16 and going clockwise), from the large linear seki in the middle and the seki in the lower right corner (with its encoding starting at q1 and going clockwise). The big seki in the middle of the board is topologically a linear seki, starting with the black eye at a11 (therefore its encoding between both ; signs starts with a 0) and ending with the white eye at t9.

The second example in Diagram 32 has the encoding $0(2\underline{2})^3 1(\underline{2}2)^5;0(2\underline{2})^8 1\underline{1}(2\underline{2})^7 1(\underline{2}2)^5;$ $1\underline{2}2\underline{2}1\underline{1}$. The first part before the first ; sign encodes the linear seki starting with an eye at

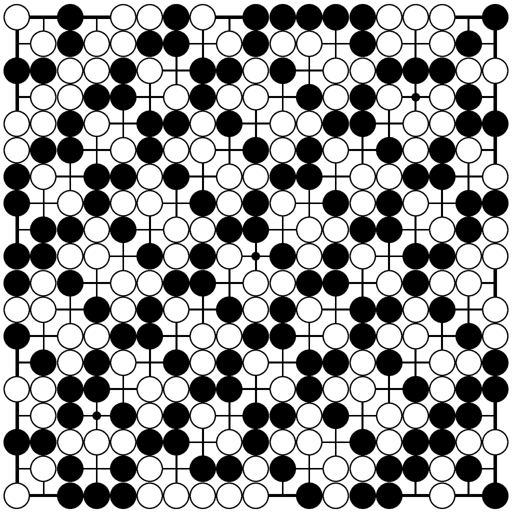

Diagram 31.
A full board seki of G. Hungerink

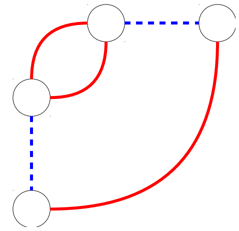

Figure 9: The upper left corner and the colour switched lower right corner of Diagram 31 as BSG with encoding $\underline{1}2\underline{1}1$.

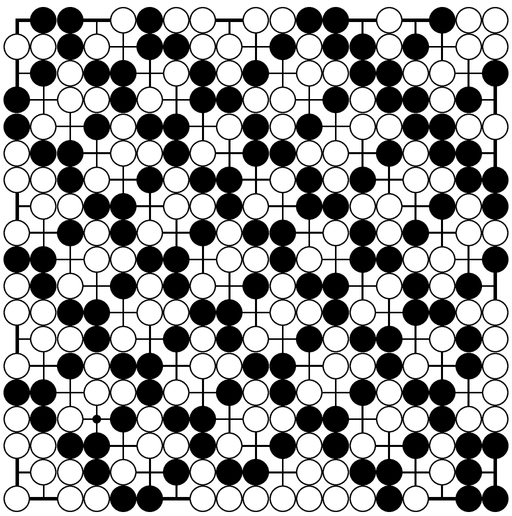

Diagram 32.
Another large seki of G. Hungerink covering the whole board except the lower left corner

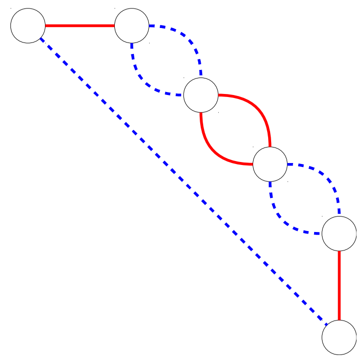

Figure 10: The upper right corner of Diagram 32 as BSG with encoding $1\underline{2}2\underline{2}1\underline{1}$.

a7 and ending with an eye at a12. The part between the two ; signs encodes the linear seki that starts with a black eye in the lower right corner and ends with a white eye at j19. The seki in the upper right corner is circular. Its encoding follows the second ; sign and encodes the seki clockwise starting at o19.

## 5.2 Cutting off a Single Stone from a Chain

If the local $2 \times 3$ pattern in Diagram 33 is part of a Basic Seki and if the liberty underneath ◬ does not have other (non-visible) chains as neighbours then this pattern can be replaced by the one in Diagram 34. This exchange works of course as well if both pattern are rotated, mirrored, if colours are switched and may also work if the stone ◬ is black.

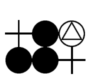

Diagram 33.

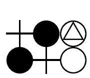

Diagram 34.

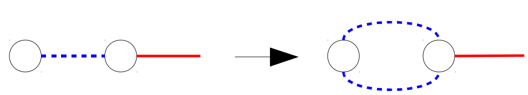

Figure 11: The change of BSG

For example, Diagram 35 can be deformed into Diagram 36 which after replacing pattern 33 by 34 gives Diagram 37.

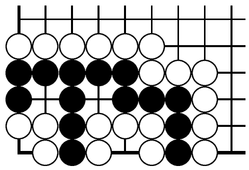

Diagram 35.
with encoding $01\underline{1}1$

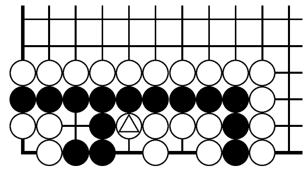

Diagram 36.
with encoding $01\underline{1}1$

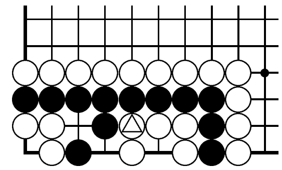

Diagram 37.
with encoding $01\underline{2}1$

In the next example (Diagrams 38, 39, 40) the left liberty in this pattern is located inside an eye. Diagram 38 is deformed into 39 which after the exchange of pattern turns into Diagram 40.

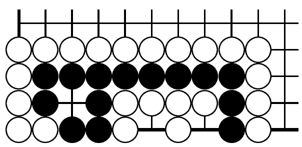

Diagram 38.
with encoding $0\underline{1}1$

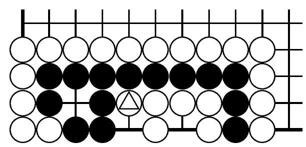

Diagram 39.
with encoding $0\underline{1}1$

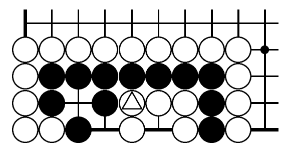

Diagram 40.
with encoding $0\underline{2}1$

In the next two examples, the exchange of pattern turns Diagram 41 into 42 and Diagram 43 into 44.

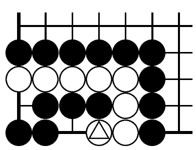

Diagram 41.
with encoding $\underline{1}1$

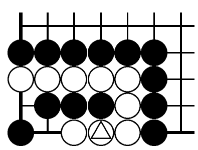

Diagram 42.
with encoding $\underline{2}1$

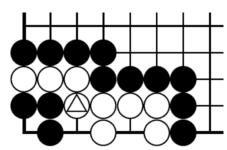

Diagram 43.
with encoding $0\underline{1}1$

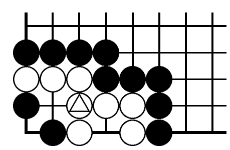

Diagram 44.
with encoding $0\underline{2}1$

In the next example a slight deformation of Diagram 45 into 46 and an exchange of pattern and a minor deformation leads to Diagram 47.

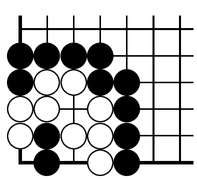

Diagram 45.
with encoding $\underline{1}11$

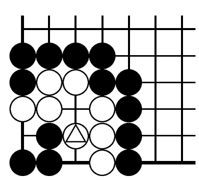

Diagram 46.
with encoding $\underline{1}11$

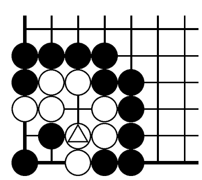

Diagram 47.
with encoding $\underline{2}11$

## 5.3 Creating an Eye

If the white chain in Diagram 48 is not part of an eye then introducing a 2-chain eye like in Diagram 49 or 50 increases the number of white chains by one and adds one white liberty so that both white chains still have two liberties afterwards. This exchange of pattern may require a deformation to create the space for the eye.

A variation of this eye-creation on the edge of the board changes Diagrams 51 and 52 into 53. Removing an eye from Diagrams 51 and 52 (and deforming the black chain to make it too large to be further extended and sacrificed by Black) to get Diagram 54 also generates a seki position.

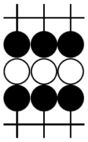
Diagram 48.

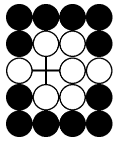
Diagram 49.

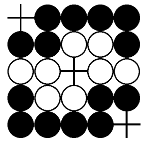
Diagram 50.

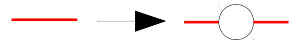
Figure 12:
The change of BSG

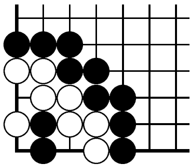
Diagram 51.
with encoding $\underline{1}11$

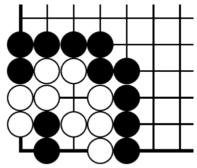
Diagram 52.
with encoding $\underline{1}11$

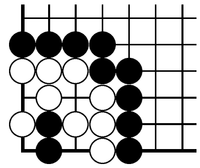
Diagram 53.
with encoding $\underline{1}111$

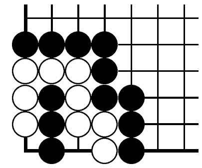
Diagram 54.
with encoding $\underline{1}1$

The addition of a 2-chain eye changes a 2-chain seki in Diagram 55 into a 3-chain seki in Diagram 56.

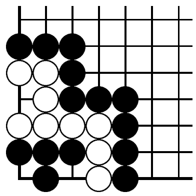
Diagram 55.
with encoding $0\underline{1}1$

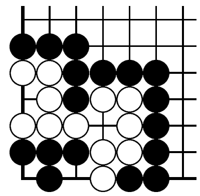
Diagram 56.
with encoding $0\underline{1}11$

Instead of creating a 1-point eye by adding one chain one can also create a 2-point eye by adding two chains. The change from Diagram 57 to Diagram 58 can be interpreted as executing the above extension twice.

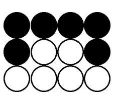
Diagram 57.

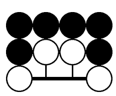
Diagram 58.


Figure 13: The change of BSG

In the change from Diagram 59 to Diagram 60 even 3 chains are added.

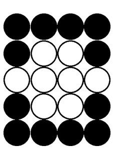
Diagram 59.

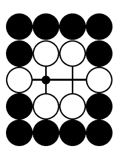
Diagram 60.

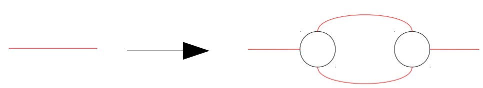
Figure 14: The change of BSG

Diagrams 61 and 62 show at their lower edge examples for adding a white 2-point eye and 2 or 3 white chains.

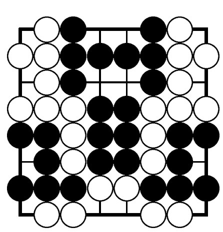
Diagram 61.
with encoding $0\underline{1}111\underline{1}$

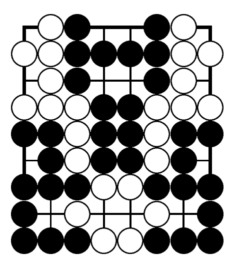
Diagram 62.
with encoding $0\underline{1}121\underline{1}$

## 5.4 Bamboo Joints in Basic Seki

If there is a bamboo joint within a Basic Seki linking, say, two white chains like in Diagram 63 then both chains can not have further liberties because each chain has only two liberties in a Basic Seki. Furthermore, to have a seki at least one black chain has to be attached to one of the liberties, like in Diagram 64. If Black makes a move on that liberty A then the white chains come under atari and to be a seki, Black itself must come under atari, which is only possible if the other liberty B of the bamboo joint is also a liberty of the same black chain and thus B is the only liberty of Black after making a move on A. The situation must therefore be like in Diagrams 65 and 68 with one and the same black chain being the neighbour to both liberties A and B.

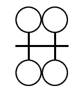

Diagram 63.

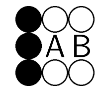


Diagram 64.

By stretching the bamboo joint and inserting one or more black stones the positions in Diagrams 66, 67 result from 65 and 69, 70 result from 68.

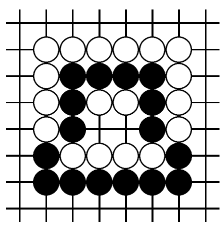

Diagram 65.
with encoding $2\underline{1}$

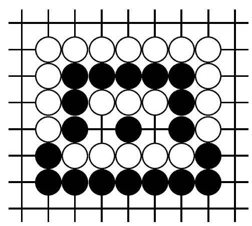

Diagram 66.
with encoding $2\underline{2}$

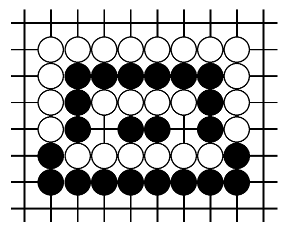

Diagram 67.
with encoding $2\underline{2}$

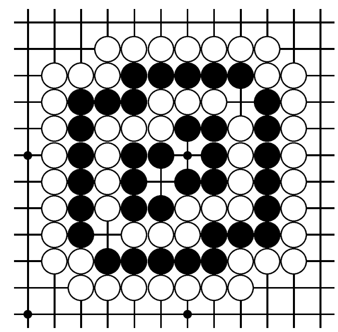

Diagram 71.
with encoding $2\underline{2}$

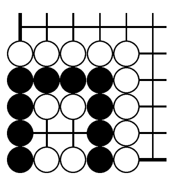

Diagram 68.
with encoding $2\underline{1}$

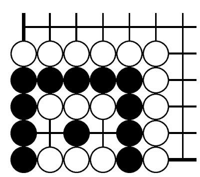

Diagram 69.
with encoding $2\underline{2}$

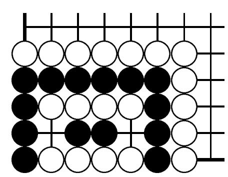

Diagram 70.
with encoding $2\underline{2}$

The BSG and its numerical encoding are fully equivalent except for the case that a node has two white and two black edges. In this case the neighbouring stones of the corresponding liberty on the board could be black, white, black, white (BWBW or equivalently WBWB) when surrounding the liberty, like in Diagrams 66, 67 69, 70, or they could be black, white, white, black (BWWB or equivalently BBWW) like in Diagram 71.

In both cases the numerical encoding is $2\underline{2}$ but one could put slightly more information in the graph by choosing the graph 2+2 h) in Figure 15 in the appendix for Diagram 70 and choosing the graph 2+2 g) for Diagram 71. This distinction between the two types of nodes of degree 4 does not matter topologically, only when one wants to realize such positions on a board. The BBWW type requires more space because at least one pair of chains (BB or WW) need to be separated by a living group of opposite colour like in Diagram 71.

# 6 Summary

Seki in which each chain is essential and has exactly 2 liberties (except negligible chains of at most two stones, with only one liberty located within opponent 1-chain 1-liberty eyes) are called Basic Seki in this paper.

After describing deformations of seki that do not change their topology, a special type of graph (BSG) is developed that displays this topology. Each node of such a graph represents a liberty and each edge in the graph represents a chain of the seki.

It is shown that a Basic Seki has either only 2 empty points or that each node in the graph can have at most 2 neighbouring nodes and as a consequence that Basic Seki can have only a linear or a circular topology. Moreover, in the case of $> 2$ empty points, if 2 of them are the liberties of different chains then these chains need to have the same colour for the position to be a seki. This allows to recognize Basic Seki locally without having to play out moves throughout the whole seki.

It also enabled the development of a numerical encodings of Basic Seki which in turn made it possible to generate all topological types of Basic Seki.

In the last section the attachment of seki and two techniques for complicating Basic Sekis are described and illustrated with examples and their numerical encoding.


# Acknowledgement

The author would like to thank the referees for their comments – especially Harry Fearnley, for his many suggestions.


# References


[1] Harry Fearnley: Shared Life – an Overview `http://harryfearnley.com/go/seki/overview/overview_full.doc`

[2] T. Wolf: A Classification of Semeai with Approach Moves, ICGA Journal, Vol. 38, No. 4 (Dec 2015), pp. 224-241. `http://lie.math.brocku.ca/twolf/papers/semeai.pdf`

[3] Thore Graepel, Mike Goutrié, Marco Krüger, Ralf Herbrich: Learning on Graphs in the Game of Go, Artificial Neural Networks ICANN 2001, Volume 2130 of the series Lecture Notes in Computer Science pp 347-352 and `http://www.herbrich.me/papers/go.pdf`

[4] T. Wolf: The Mathematics of Seki, talk given at the 2013 ACGA Spring Go Expo, 23-24 March at Harvard University, Cambridge, Massachusetts. `http://lie.math.brocku.ca/twolf/papers/sekitalk2.pdf`

[5] Ger Hungerink, in "Go" the Journal of the Duch Go Association, vol 14 (1976/77), issue 5.

[6] Sensei's Library `http://senseis.xmp.net/?Seki`

[7] The On-Line Encyclopedia of Integer Sequences (OEIS) `https://oeis.org/A006356`


# Appendix

The following figure shows all Basic Seki Graphs with up to 4 edges (chains) and their numerical encoding.

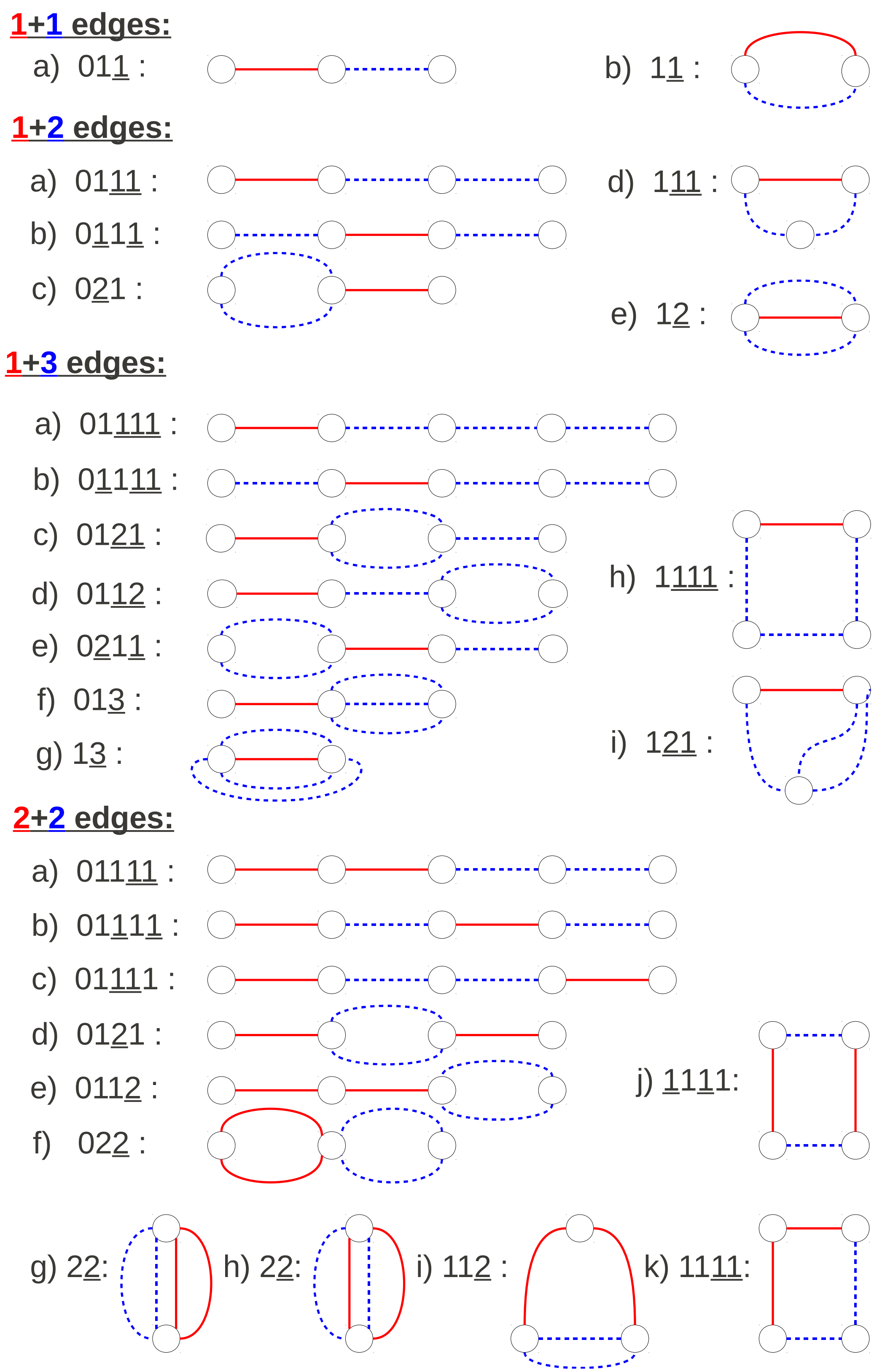


Figure 15: Basic Seki Graphs (with max 4 edges) and their encoding