Bifurcations of local structure maps as predictors of phase transitions in asynchronous cellular automata

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Abstract. We show that the local structure approximation of sufficiently high order can predict the existence of second order phase transitions belonging to the directed percolation university class in α-asynchronous cellular automata.

Probabilistic cellular automata (PCA) rules known as *α-asynchronous rules* are obtained by a random perturbation of the deterministic updating rule: instead of updating all cells simultaneously, we update each cell independently with probability α, the *synchrony rate*, and leave its state unchanged with probability $1-\alpha$.

The systematic exploration of the properties of α-asynchronous Elementary Cellular Automata by numerical simulations [2] identified rules which exhibited a qualitative change of behaviour for a continuous variation of the synchrony rate: there exists a critical value of α_c which separates an *active phase* in which the system fluctuates around an equilibrium and an *absorbing phase* where the system is rapidly attracted towards a fixed point where all cells are in the same state.

Using the techniques from statistical physics, this abrupt change of behaviour was then identified as a second order phase transitions which belong to the directed percolation (DP) universality class [1]. This identification was conducted by taking as an order parameter the density, that is the average number of cells in state 1, and, up to symmetries, nine rules were found to exhibit such DP behaviour. Their Wolfram numbers are 6, 18, 26, 38, 50, 58, 106, 134, and 146.

The aim of this paper is to study to which extent this second order phase transition can be predicted with analytical techniques. We are in particular interested in answering two questions: (a) Can we explain the existence of the two active and absorbing phases? (b) Can we propose an approximation of the value of the critical synchrony rate α_c that separates the two phases?

Our approach is based on so-called *local structure theory*, proposed in 1987 by H. A. Gutowitz *et al.* [5] as a generalization of the mean-field theory for cellular automata. The basic idea of this theory is to consider probabilities of blocks

(words) of length k and to construct a map on these block probabilities, which, when iterated, approximates probabilities of occurrence of the same blocks in the actual orbit of a given cellular automaton. In the case of nearest-neighbour binary rule, the aforementioned map is 2^k-dimensional, where k is called the *level* of local structure approximation. However, using the method proposed in [3], it can be reduced to equivalent, but somewhat simpler 2^{k-1} dimensional map.

We will assume that the dynamics takes place on a one-dimensional lattice. Let $s_i(t) \in \{0,1\}$ denotes the state of the lattice site i at time t, where $i \in \mathbb{Z}$, $t \in \mathbb{N}$. We will say that the site i is occupied (empty) at time t if $s_i(t) = 1$ (resp., $s_i(t) = 0$). *A Deterministic elementary cellular automaton* is a dynamical system governed by the local function $f : \{0,1\}^3 \to \{0,1\}$ such that

$$s_i(t+1) = f\big(s_{i-1}(t), s_i(t), s_{i+1}(t)\big),$$

for all $i \in \mathbb{Z}$ and for all $t \in \mathbb{N}$. Function f is to be called a *rule* of CA.

In a probabilistic cellular automaton, lattice sites simultaneously change states from 0 to 1 or from 1 to 0 with probabilities depending on states of local neighbours. A common method for defining PCA is to specify a set of local transition probabilities. For example, in order to define a nearest-neighbour PCA one has to specify the probability $w(s_i(t+1))|s_{i-1}(t), s_i(t), s_{i+1}(t))$ that the site $s_i(t)$ with nearest neighbors $s_{i-1}(t), s_{i+1}(t)$ changes its state to $s_i(t+1)$ in a single time step.

We will now define *α-asynchronous elementary cellular automata*. Let $\alpha \in [0,1]$ and let f be a local function of some deterministic CA with Wolfram number $W(f)$. Corresponding α-asynchronous elementary cellular automaton with rule number $W(f)$ is a probabilistic CA for which transition probabilities are

$$w(1|x_1x_2x_3) = \alpha f(x_1, x_2, x_3) + (1-\alpha)x_2. \tag{1}$$

Let us denote by $P_t(\mathbf{b})$ the probability of occurrences of blocks $\mathbf{b} = b_1b_2\dots b_n$ after t iterations of the PCA rule, where $\mathbf{b} \in \{0,1\}^*$. These probabilities, to be called block probabilities, form an infinite hierarchy that we can arrange by defining $\mathbf{P}_t^{(k)}$ as a column vector that holds all the k-block probabilities sorted in lexical order.

Let us now suppose that a PCA is given, and we know its transition probabilities w. Local structure map associated with that PCA is then given by

$$\begin{aligned} P_{t+1}(a_1\dots a_k) = &\sum_{b\in\{0,1\}^{k+2}} \prod_{i=1}^{k} w(a_i|b_ib_{i+1}b_{i+2}) \\ &\times \frac{P_t(b_1\dots b_k)P_t(b_2\dots b_{k+1})P_t(b_3\dots b_{k+2})}{\big(P_t(b_2\dots b_k0)+P_t(b_2\dots b_k1)\big)\big(P_t(b_3\dots b_{k+1}0)+P_t(b_3\dots b_{k+1}1)\big)}. \end{aligned} \tag{2}$$

We refer the reader to [3] for details of its construction. It should be understood that the above is a system of 2^k equations, so that we have a separate equation for each $(a_1\dots a_k) \in \{0,1\}^k$. In vector form we will write

$$\mathbf{P}_{t+1}^{(k)} = \Lambda^{(k)}\left(\mathbf{P}_t^{(k)}\right), \tag{3}$$

where $\Lambda^{(k)}$, defined by eq. (2), will be called *local structure map* of level k.

Not all components of the block probability vector $\mathbf{P}_t^{(k)}$ are independent. This is due to so-called consistency conditions, that is, equations of the type $P_t(a_1 \dots a_{k-1}0)+P_t(a_1 \dots a_{k-1}1) = P_t(a_1 \dots a_k)$. By exploiting these conditions, 2^k-dimensional local structure map of level k can be reduced to equivalent, but somewhat simpler, 2^{k-1} dimensional map [3].

Detailed analysis of reduced local structure maps for α asynchronous PCA belonging to DP universality class reveals that these maps exhibit transcritical bifurcations as α changes. The bifurcation does not necessarily happens for all k, but rather appears when k is sufficiently high. For rules 6, 18, 38, 50, 106, and 134, we were able to compute fixed points of local structure maps of level 3 directly, by solving equations $\mathbf{P}^{(3)} = \Lambda^{(3)}\left(\mathbf{P}^{(3)}\right)$ with he help of computer algebra software (the maps were transformed to reduced form first). We were also able to determine the stability of these fixed points. Details of these calculations can be found in [4]. Here we only present graphs of typical bifurcation diagrams obtained that way, as shown in Figure 1. Vertical axis in these diagrams corresponds to $P(1)$, which can be obtained from $\mathbf{P}^{(3)}$ by using consistency conditions, as $P(1) = P(100) + P(101) + P(110) + P(111)$. In all three diagrams, the absorbing fixed point $P(1) = 0$ is present, shown as the horizontal line. The active fixed point is represented by the smooth curve, partially solid (stable) and partially dashed (unstable). Exchange of stability takes place at the bifurcation point (circled).

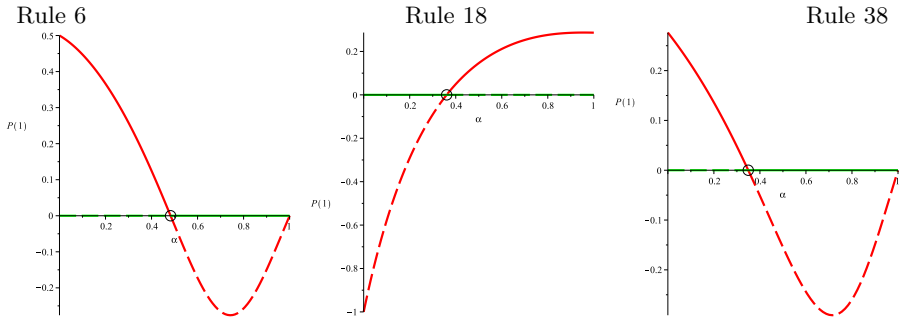

Fig. 1. Bifurcation diagram for local structure equations of level three for rules 6, 18, and 38.

For the three remaining rules, 26, 58, and 146, the local structure map $\Lambda^{(3)}$ does not exhibit a transcritical bifurcation, so it is necessary to consider higher order maps, of level four (for rules 26 and 146) and five (for rule 58). Absorbing fixed points of these maps have the same structure as previously described, with $P(1) = 0$. Unfortunately, equations for their active fixed points cannot be solved even with the help of symbolic algebra software, due to the size of relevant

equations. It is, however, possible to find the stable branch of the bifurcation diagram by iterating these maps many times, so they converge sufficiently close to the stable fixed point. We performed such iterations for all three cases, and the results are shown in Figure 2. Even though the unstable branch of the active fixed point is missing, it is evident that the active phase appears abruptly as α increases, which provides a strong evidence for transcritical bifurcation.

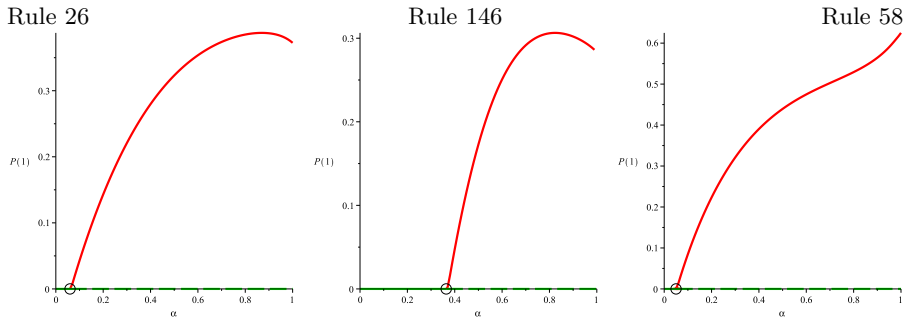

Fig. 2. Partial bifurcation diagrams for local structure equations of level 4 rules 26 and 146, and level 5 for rule 58. Diagrams were obtained numerically with 10^5 iterations.

In summary, we can say that the local structure approximation of order 3 to 5 can predict existence of the phase transition for all DP rules. The local structure map for each of these rules exhibits a transcritical bifurcations, and the direction of the bifurcation agrees with the direction of the phase transition observed experimentally, that is, if the active phase appears (disappears) as α increases, then the non-zero fixed point of the local structure map becomes stable (unstable) as α increases. The point at which the transcritical bifurcation occurs is, however, rather far from the critical point observed experimentally.

Can this be improved by increasing the order of the local structure approximation? The answer is indeed yes, although we cannot expect to be able to find explicit symbolic expressions for fixed points of eq. (2) when k is large. One can, however, iterate $\Lambda^{(k)}$ many times, starting from some generic initial condition, and when this is done, the orbit of $\Lambda^{(k)}$ indeed converges to a stable fixed point, which, depending on the value of α, can be zero or non-zero.

We performed iterations of $\Lambda^{(k)}$ maps for $k = 2 \ldots 9$ for all DP rules, and plotted $P_t(1)$ as a function of α after $t = 10^4$ iterations. Results are shown in Figure 3, together with curves obtained "experimentally" by iterating a given rule for 10^5 steps, using randomly generated initial configurations with $4{\cdot}10^4$ sites and periodic boundary conditions. One can clearly see that local structure maps not only predict existence of phase transitions, but also seem to approximate behaviour of $P_t(1)$ vs. α curves with increasing accuracy as the order of the approximation increases.

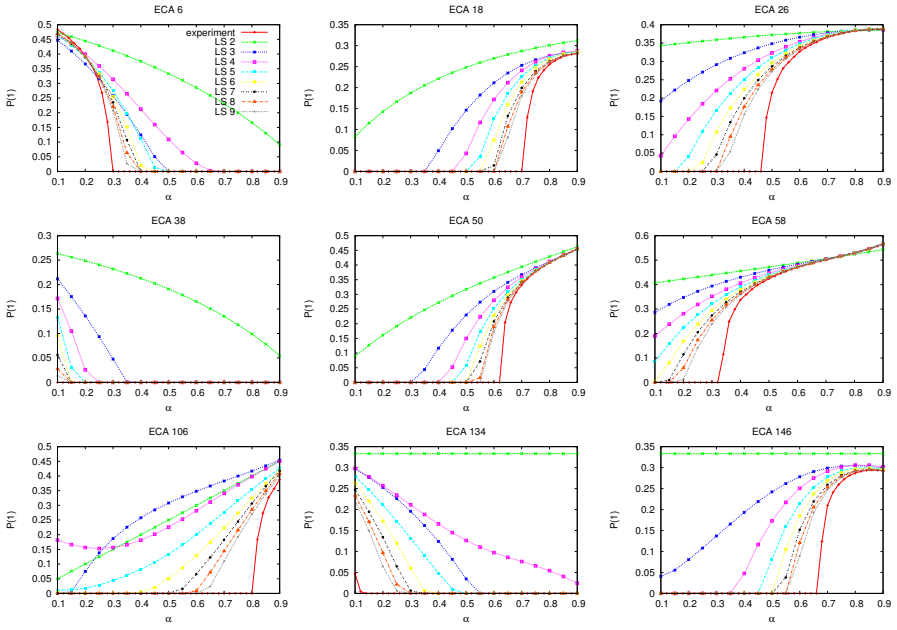

Fig. 3. Experimental results together with local structure approximations up to level 9.

Based on the evidence presented in this paper, we suspect that the same may be true for other probabilistic CA rules belonging to DP universality class. It is already known to be true for the probabilistic mixture of rules 182 and 200 studied by Mendonça and de Oliveira [6]. We plan to investigate this conjecture for other PCA rules.

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